$$+\varepsilon^2\left(\frac{3}{5}a^2\varphi''u_5+\frac{5}{4}ab\varphi''u_4+\frac{1}{10}ab\varphi''\frac{u_2u_3}{u_1}-\frac{1}{20}ab\varphi''\frac{u_2^3}{u_1^2}+\frac{2}{3}b^2\varphi''u_3+\right.$$
$$+\frac{9}{5}a^2\varphi'''u_1u_4+3a^2\varphi'''u_2u_3+\frac{7}{2}ab\varphi'''u_1u_3+\frac{23}{10}ab\varphi'''u_2^2+$$
$$+\frac{5}{3}b^2\varphi'''u_1u_2+\frac{23}{10}a^2\varphi^{IV}u_1^2u_3+\frac{31}{10}a^2\varphi^{IV}u_1u_2^2+\frac{15}{4}ab\varphi^{IV}u_1^2u_2+$$
$$\left.+\frac{1}{2}b^2\varphi^{IV}u_1^3+\frac{8}{5}a^2\varphi^{V}u_1^3u_2+\frac{1}{2}ab\varphi^{V}u_1^4+\frac{1}{8}a^2\varphi^{VI}u_1^5\right)+\ldots$$

From this expression and from analysis of the transfer formula (2.7) we see how the Lie–Becklund symmetry for the Korteweg–de Vries and Burgers equations transforms into a formal symmetry (7.2) for the equation (7.1) that does not satisfy the cutoff condition.

LITERATURE CITED

1. L. V. Ovsyannikov, Group Properties of Differential Equations [in Russian], Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1962).
2. L. V. Ovsyannikov, "Group properties of the equations of mechanics," in: Continuum Mechanics and Related Problems [in Russian], Nauka, Moscow (1972).
3. V. A. Baikov, R. K. Gazizov, and N. Kh. Ibragimov, "Approximate symmetry," Mat. Sb., 136, No. 4 (1988).
4. V. A. Baikov, R. K. Gazizov, and N. Kh. Ibragimov, "Approximate group analysis of the nonlinear equation $u_{tt} - (f(u)u_x)_x + \varepsilon\varphi(u)u_t = 0$," Differents. Uravn., No. 7 (1988).
5. N. Kh. Ibragimov, Groups in Mathematical Physics [in Russian], Nauka, Moscow (1983).
6. R. R. Rosales, "Exact solutions of some nonlinear evolution equations," Stud. Appl. Math., 59, 117 (1978).
7. M. Flato, G. Pinczon, and J. Simon, "Nonlinear representations of Lie groups," Ann. Scient. Ecole Norm. Sup. Paris, 10, No. 4 (1977).
8. E. Taflin, "Analytic linearization of the Korteweg–de Vries equation," Pacif. J. Math., 108, No. 1 (1983).

QUANTITATIVE CHARACTERISTICS OF THE MAIN CONCEPTS OF LINEAR CONTROL THEORY

S. K. Godunov

UDC 519.6

In classical linear control theory there is detailed study of the possibility of selecting a control u(t) which would make it possible to obtain some optimum behavior of trajectory x(t) described by the system

$$\frac{d}{dt}x(t)=Ax(t)+Bu(t). \tag{1}$$

Normally it is assumed to be possible to obtain information about the behavior of this trajectory only from the vector of observation z(t) = Cx(t). We limit ourselves to considering a particular, but important in many typical cases, independent of time t, matrix A, B, C. There is extensive use (see, e.g., [1-3]) of the concept of controllability for pair A, B and the dual concept to each other of stabilizability for A, B and detectability for pair A, C. (If A, B is controllable or stabilizable, then A*, B* is observable and detectable, and conversely).

We introduce criteria (necessary and sufficient) for controllability and stabilizability. Pair A, B is controllable if the linear shell of columns for the composite matrix

$$(B \vdots AB \vdots A^2B \vdots \ldots \vdots A^{N-1}B) \tag{2}$$

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 49-52, March-April, 1989. Original article submitted July 29, 1988.

has the maximum possible rank N. Here N is the dimension of the space in which the operator acts, and $N \times N$ is prescribed for the matrix of A. Pair A, B is stabilizable if the linear shell of the matrix columns (2) contains all of the invariant root subspaces relating to points of the spectrum not strictly lying on the left-hand half-plane. A consequence of these criteria is the statement: pair A, B is controllable when, and only when both pairs A, B and −A, B are stabilizable. These facts make it possible in discussing the question of developing quantitative characteristics for the degree of controllability, stabilizability, observability, and detectability to limit the choice only to the problem of stabilizability.

It is evident that possible controls Bu(t) are in essence only determined by the subspace conforming with the linear shell of vectors whose coordinates form column B. A vector for the subspace is presented in the form $Bu = B(B^*B)^{-1}B^*\lambda$ [λ is N-dimensional vector; $B(B^*B)^{-1}B^*$ is orthogonal projector]. If computation of the projector for the prescribed matrix of B causes difficulty, which develops in the cases of poor specification of B, then this indicates that in this case the space for control vectors Bu may change markedly with unsubstantial changes in B, i.e., it indicates that the space for control vectors will not be determined reliably. We limit ourselves to studying stabilizability with firmly prescribed description of possible controls when projector $B(B^*B)^{-1}B^*$ is assumed to be known or calculable without any difficulty.

In addition, it is assumed that the unit of measurement for time t in the model system, for the study of which it is concluded that it is possible to regard the pair A, B as stabilizable, is selected so that A converts to a matrix of a single norm. In other words, instead of the original system (1) we suggest consideration of the system $(d/d\tau)y(\tau) = A_0y(\tau) + Bv(\tau)$, in which

$$\tau = \|A\|t, \quad A_0 = \frac{1}{\|A\|}A, \quad v(\tau) = \frac{1}{\|A\|}u(\|A\|t), \quad y(\tau) = x(\|A\|t).$$

Clarification of the question of whether the pairs A, B or $A_0 = (1/\|A\|)A$, B are stabilizable is reduced (see [1-3]) to checking whether it is possible for each N-dimensional vector φ to determine $v(\tau)$ so that solution of the Cauchy problem $dy(\tau)/d\tau = A_0y(\tau) + Bv(\tau)$, $y(0) = \varphi$ has a finite integral

$$\int_0^\infty (\|y(\tau)\|^2 + \|Bv(\tau)\|^2)\,d\tau = \frac{1}{\|A\|}\int_0^\infty (\|A\|^2\|x(t)\|^2 + \|Bu(t)\|^2)\,dt < \infty. \tag{3}$$

If the system is stabilizable, then a single control $Bv(\tau)$ exists with which this integral takes the minimum possible value. In order to construct this equation it is necessary to construct a solution for the Hamiltonian system

$$\frac{d}{d\tau}\begin{pmatrix} y(\tau) \\ \lambda(\tau)\end{pmatrix} = \begin{pmatrix} \frac{1}{\|A\|}A & B(B^*B)^{-1}B^* \\ I_N & -\frac{1}{\|A\|}A^* \end{pmatrix}\begin{pmatrix} y(\tau) \\ \lambda(\tau)\end{pmatrix}, \tag{4}$$

having a prescribed value $y(0) = \varphi$, and to assume that $Bv(\tau) = B(B^*B)^{-1}B^*\lambda(\tau)$. In the case of stabilizability for A, B this solution exists and it is unique if it satisfies the requirement $\|y(\tau)\| \to 0$, $\|\lambda(\tau)\| \to 0$ with $\tau \to \infty$. The reverse assertion is also correct. From existence of a solution for the boundary problem now stated the pair A, B should be stabilizable.

As is well known [4], the spectrum of the Hamiltonian matrix

$$\mathscr{H} = \begin{pmatrix} \frac{1}{\|A\|}A & B(B^*B)^{-1}B^* \\ I_N & -\frac{1}{\|A\|}A^* \end{pmatrix},$$

i.e., a set of roots for the equation $\det(\mathscr{H} - \mu I_{2N}) = 0$ placed symmetrically in relation to the origin. For each characteristic root μ_j it is necessary to find a symmetrical root $\mu_k = -\mu_j$ with the same multiplicity. The possibility of finding a decreasing solution $y(\tau)$, $\lambda(\tau)$ with any N-dimensional $y(0) = \varphi$ means that the number of roots lying strictly in the left-hand half-plane should not be less than N. From the Poincaré theorem provided above for symmetry of the Hamiltonian matrix spectrum it emerges that for stabilizability absence of purely imaginary characteristic roots for $\mathscr{H}$ is necessary, or what amounts to the same, finiteness of parameter $\varkappa(\mathscr{H})$, characterizing the "nature of dichotomy" of the $\mathscr{H}$ spectrum by an imaginary axis. This parameter was introduced in [5, 6], where there was determination as a norm of $\varkappa(\mathscr{H}) = \|H\|$ of the Hermitian positive-definite matrix H computed in terms of $\mathscr{H}$ by one of the following integrals:

$$H = \frac{1}{2\pi}\int_{-\infty}^{+\infty} [\mathscr{H}^* + itI_{2N}]^{-1}[\mathscr{H} - itI_{2N}]^{-1}\,dt = \int_{-\infty}^{+\infty} G^*(t)\,G(t)\,dt.$$

Here G(t) is a Green matrix, i.e., a limited solution of the matrix differential equation $\frac{d}{dt}G(t) = \mathscr{H}G(t) + \delta(t)I_{2N}$ with $-\infty < t < +\infty$.

It was shown in [5-7] that calculation of projectors in invariant space $\mathscr{H}$ relating to parts of the spectrum in the left- and right-hand complex half-space is reduced to calculating G(+0), −G(−0), respectively, which occur with these projectors. The rate of convergence of iterations in the computation process is governed by the value of $\varkappa(\mathscr{H})$, and the convergence is more rapid, the lower $\varkappa$. With very large values of this parameter it is natural to abandon computation by stating for practical purposes existence of dichotomy for the spectrum of $\mathscr{H}$. It is extremely important that the level of stability G(±0) in relation to small perturbations of $\mathscr{H}$ is also estimated in terms of $\varkappa(\mathscr{H})$. Finiteness for $\varkappa(\mathscr{H})$ is a necessary condition for stabilizability of pair A, B.

Any decreasing trajectory y(τ), λ(τ) with τ → +∞, i.e., a solution of system (4), is presented with τ > 0 in the form

$$\begin{pmatrix} y(\tau) \\ \lambda(\tau) \end{pmatrix} = G(\tau)\begin{pmatrix} y(0) \\ \lambda(0) \end{pmatrix} = e^{\tau\mathscr{H}}G(+0)\begin{pmatrix} y(0) \\ \lambda(0) \end{pmatrix},$$

where λ(0) is found for prescribed y(0) = φ as a solution of the vector equation

$$G(-0)\begin{pmatrix} \varphi \\ \lambda(0) \end{pmatrix} = 0. \tag{5}$$

It is noted (see, e.g., [5, 7]) that the process of computing projectors G(+0), −G(−0) may be supplemented by the uncomplicated procedure of calculating H and $\varkappa(\mathscr{H}) = \|H\|$.

The possibility of unambiguously determining Eq. (5) by expressing λ(0) in terms of φ

$$\lambda(0) = K\varphi, \tag{6}$$

together with finiteness for parameter $\varkappa(\mathscr{H})$, which provides the possibility of computing G(±0), is a necessary and sufficient condition for stabilizability of pair A, B. Norm $\sqrt{1 + \|K\|^2}$ of representation

$$\varphi \to \begin{pmatrix} y(0) \\ \lambda(0) \end{pmatrix} = \begin{pmatrix} \varphi \\ K\varphi \end{pmatrix} = \begin{pmatrix} I_N \\ K \end{pmatrix}\varphi,$$

which will be assumed to be infinite if λ(0) cannot be found unambiguously from (5) even with certain φ, cannot be considered as a stability characteristic for the solution y(0), λ(0) in relation to small perturbations of φ. The optimum control minimizing integral (3) is given by reverse constraint equations

$$\lambda(\tau) = Ky(\tau),\ Bv(\tau) = B(B^*B)^{-1}B^*y(\tau).$$

Our proposal consists of constructing a value from $\varkappa(\mathscr{H})$, $\sqrt{1 + \|K\|^2}$ whose finiteness provides finiteness for these two parameters and, consequently, the feasibility of constructing a stabilizing control. We designate this value, a stabilizability characteristic, in terms of Stab [A, B].

For example, by assuming that

$$\mathrm{Stab}\,[A, B] = \frac{100}{\max\{\sqrt{\varkappa(\mathscr{H})},\ \sqrt{1+\|K\|^2}\}}$$

and making use of the fact that $\varkappa(\mathscr{H}) \geq 1$, $1 + \|K\|^2 \geq 1$, we shall always have Stab [A, B] $\leq$ 100 characterizing the "degree of stabilizability" as if it were as a percentage. Other proposals are also possible for the form of the equation expressing Stab [A, B] in terms of $\varkappa(\mathscr{H})$ and $\|K\|$.

The necessity of introducing numerical characteristics for the degree of stabilizability, controllability, detectability, and observability became clear in the process of analyzing the set of equations by means of numerical methods of linear algebra giving a result with a guaranteed estimate of accuracy. A review of problems arising in developing these methods has been given in [6].

The author thanks A. Ya. Bulgarkov and V. M. Gordienko for discussions during which the reasoning given in the present work arose.

LITERATURE CITED

1. Uonem, Linear Multidimensional Control Systems [in Russian], Nauka, Moscow (1980).
2. A. A. Krasovskii (ed.), Handbook for Automatic Control Theory [in Russian], Nauka, Moscow (1987).
3. A. A. Pervozvanskii, Course in Automatic Control Theory [in Russian], Nauka, Moscow (1986).
4. V. A. Yakubovich and V. M. Starzhinskii, Parametric Resonance in Linear Systems [in Russian], Nauka, Moscow (1987).
5. S. K. Godunov, "The problem of dichotomy for the spectrum of a matrix," Sib. Mat. Zh., 27, No. 5 (1986).
6. S. K. Godunov, "Problem of guaranteeing accuracy in numerical methods of linear algebra," Proc. Int. Cong. Mathematicians, Vol. 2, Berkeley, CA (1986).
7. A. N. Malyshev, "Computation of invariant subspaces of the regular linear bundle of a matrix," Preprint, Inst. Mat. Sib. Otd. Akad. Nauk SSSR, No. 6, Novosibirsk (1988).

ASYMPTOTICS OF A VELOCITY FIELD AT CONSIDERABLE DISTANCES FROM A SELF-PROPELLED BODY

V. V. Pukhnachev

UDC 532.516

Stationary flow is considered for a viscous incompressible liquid outside a finite body in a three-dimensional space. Velocity distribution is prescribed at the surface of the body for a liquid with zero overall flow rate over this surface. At infinity the velocity vector tends toward a zero constant vector. External mass forces may act on the liquid decreasing quite rapidly with distance from the body. It is required that the total pulse applied to the liquid by the boundary of the body and by mass forces equals zero. The conditions listed form a boundary problem for Navier–Stokes equations which we call the problem of pulse-free flow or the problem of flow around a self-propelled body. Asymptotics are constructed for the solution of this problem at considerable distances from the body assuming that this solution exists. These asymptotics have marked differences from those for solving the classical problem of flow around a towed body [1-3].

1. Statement of the Problem. We formulate the problem of pulse-free flow around a body by a viscous liquid. Let Σ be a smooth closed surface in $\mathbf{R}^3$, and Ω be the external surface in relation to the Σ region. We consider in this region a stationary set of Navier–Stokes equations and the continuity

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 52-60, March-April, 1989. Original article submitted August 28, 1988.

